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This note will prove that the approximation[1] to $e \approx \left(1+9^{-4^{6\times 7}}\right)^{3^{2^{85}}}$ that uses the digits 1,2,. . ,9 each exactly once is accurate to 18,457,734,525,360,901,453,873,570 digits. The approximation can be written as $\left(1+\frac{1}{M}\right)^{M}$ for $M=9^{2^{84}}$, and the approximation works by $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$.

To compare $A=\left(1+\frac{1}{M}\right)^{M}$ to e rewrite as $A=e^{M\ln(1+1/M)}$ and power expand $M\ln(1+1/M)=1-\frac{1}{2M}+\frac{1}{3M^2}-\frac{1}{4M^3}+\frac{1}{5M^4}+\cdots$. This is an alternating series, with each successive term smaller in magnitude than the previous, so the finite sum through a positive term is larger than the infinite sum and similarly finite sum through a negative term is strictly less than the infinite sum. This gives strict inequalities

$$1-\frac{1}{2M}<M\ln(1+1/M)<1-\frac{1}{2M}+\frac{1}{3M^2}$$

Since e^x is strictly increasing, raising each part to the e^{th}power preserves inequalities, giving

$$e^{\left(1-\frac{1}{2M}\right)}<e^{M\ln(1+1/M)}=\left(1+\frac{1}{M}\right)^{M}<e^{\left(1-\frac{1}{2M}+\frac{1}{3M^2}\right)} \qquad (1)$$

Use the series expansion $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, the fact about alternating series from above, and expand the left hand side:

$$e^{\left(1-\frac{1}{2M}\right)}=e\cdot e^{-\frac{1}{2M}}=e\left(1-\frac{1}{2M}+\frac{1}{8M^2}-\cdots\right)>e\left(1-\frac{1}{2M}\right)$$

Take the right hand side of equation (1), $e\cdot e^{-\frac{1}{2M}}\cdot e^{\frac{1}{3M^2}}$, series expand last two terms. Use the alternating series trick once

$$e^{-\frac{1}{2M}}<\left(1-\frac{1}{2M}+\frac{1}{8M^2}\right)$$

and bound the other series term by term:

$$\begin{aligned}
e^{\frac{1}{3M^2}}&=1+\frac{1}{3M^2}+\frac{1}{18M^4}+\frac{1}{162M^6}+\cdots\\
&<1+\frac{1}{M^2}+\frac{1}{M^4}+\frac{1}{M^6}+\cdots\\
&=1+\frac{1}{M^2}\left(\frac{1}{1-\frac{1}{M^2}}\right)\\
&=1+\frac{1}{M^2-1}\\
&<1+\frac{2}{M^2}
\end{aligned}$$

[1] Found by in 2004 R. Sabey as reported at http://mathworld.wolfram.com/eApproximations.html

where the last inequality is valid for $M > \sqrt{2}$ (in particular the M we care about). Multiplying these last two results and extending the inequality,

$$\left(1 - \frac{1}{2M} + \frac{1}{8M^2}\right)\left(1 + \frac{2}{M^2}\right) = 1 - \frac{1}{2M} + \frac{1}{8M^2} + \frac{2}{M^2} - \frac{1}{M^3} + \frac{1}{4M^4}$$

Note $-\frac{1}{M^3} + \frac{1}{4M^4} < \frac{1}{M^2}$ for $M > 1$. Then replacing the last two terms,

$$\begin{aligned}\left(1 - \frac{1}{2M} + \frac{1}{8M^2}\right)\left(1 + \frac{2}{M^2}\right) &< 1 - \frac{1}{2M} + \frac{1}{8M^2} + \frac{2}{M^2} + \frac{1}{M^2} \\ &= 1 - \frac{1}{2M} + \frac{25}{8M^2}\end{aligned}$$

Combining all inequalities gives

$$e\left(1 - \frac{1}{2M}\right) < A < e\left(1 - \frac{1}{2M} + \frac{25}{8M^2}\right)$$

which can be rewritten as

$$\frac{e}{2M} > e - A > \frac{e}{2M} - \frac{25e}{8M^2} \qquad (2)$$

Explicitly computing precisely either side is prohibitive to impossible for $M = 9^{2^{84}}$, but computing[2] the number of decimal digits is possible, giving bounds (using $K = 18457734525360901453873569$)

$$-K > \log_{10}\frac{e}{2M} = \log_{10}e - \log_{10}2 - 2^{84}\log_{10}9 > -K - 0.7$$

Thus $10^{-K} > \frac{e}{2M}$, and $\frac{25e}{8M^2} < 10\left(\frac{e}{2M}\right)^2 < 10^{-2K+1}$. Use the methods above:

$$\begin{aligned}\frac{e}{2M} - \frac{25e}{8M^2} &> 10^{-K}10^{-0.7} - 10^{-2K+1} \\ &> 10^{-K}(0.15) - 10^{-2K+1} \\ &> 10^{-K-1}\end{aligned}$$

Combining gives $10^{-K} > e - A > 10^{-K-1}$. Thus e and A agree for the first K decimal digits past the decimal, and differ in the $K + 1^{th}$ spot after the decimal point.

This proves $\left(1 + 9^{-4^{6\times 7}}\right)^{3^{2^{85}}} \approx$ e to 18,457,734,525,360,901,453,873,570 digits (including the leading 2 in 2.718281828459045 …). Note if factorials are allowed then they can be placed after each M in the expression, allowing arbitrary accuracy.

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[2] To get 30 digit accuracy, enter N[Log[10, E] - Log[10, 2] - 2^84 Log[10, 9], 30] at http://www.wolframalpha.com