Geometric Algebra: *the* framework for geometric computations

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1 Programming geometry: unnecessarily diverse

Geometry has been well-studied in mathematics, and many techniques have been developed.

- linear algebra (especially 3D)
- differential geometry
- projective geometry
- vector calculus
- tensor analysis
- Lie algebra
- algebraic geometry
- homogeneous coordinates (especially (3+1)D)
- quaternions and complex analysis
- ...

Often several are used to solve a single application. Interfacing is an issue, both in modeling and in computation.

Q: How do they relate?
A: They are all aspects of geometric algebra!

This connection can unify and simplify geometric programs.

2 What's new in geometric algebra?

The unification is achieved by the *geometric product* (that's why it is called an algebra) plus *geometric differentiation*.

Start with a vector space V^m over scalars K (we'll take reals), with metric Q (a bilinear form $V^m \times V^m \to K$). Define a *geometric product* $A\,B$ between elements.

- linear and distributive over $+$
- associative
- scalars commute: $\alpha\,X = X\,\alpha$ for $\alpha \in K$
- a vector squares to a scalar: $\mathbf{a}\,\mathbf{a} = Q(\mathbf{a})$

This gives a 2^m-dimensional linear space with its *Clifford algebra*. The geometric product is invertible, so we can *divide by vectors*.

That's the essence, but it does not help much. We need to understand the geometric meaning of this, and develop techniques. These were discovered by Hestenes (1960s), and go back to Grassmann (1840s) and Clifford (1870s) who were among the last looking for a true geometric algebra. Real applications are very recent (1990s).

3 Towards geometrical interpretation

We can split the geometric product of two vectors into a commutative and anti-commutative part as

$$\mathbf{a}\,\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$$

where we defined the *inner product*

$$\mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}\,(\mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a})$$

and the *outer product*

$$\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}\,(\mathbf{a}\,\mathbf{b} - \mathbf{b}\,\mathbf{a}).$$

The inner product is scalar valued. Take an orthonormal basis in the $(\mathbf{a},\mathbf{b})$-plane. Let $\mathbf{a} = a\mathbf{e}_1$, and $\mathbf{b} = b\cos\phi\,\mathbf{e}_1 + b\sin\phi\,\mathbf{e}_2$.

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \tfrac{1}{2}\,a\,b\,((\mathbf{e}_1\,\mathbf{e}_1\cos\phi + \mathbf{e}_1\mathbf{e}_2\sin\phi) + (\mathbf{e}_1\,\mathbf{e}_1\cos\phi + \mathbf{e}_2\mathbf{e}_1\sin\phi))\\
&= a\,b\cos\phi\,(\mathbf{e}_1\cdot\mathbf{e}_1) + a\,b\sin\phi(\mathbf{e}_1\cdot\mathbf{e}_2)\\
&= a\,b\,\cos\phi
\end{aligned}$$

This is the familiar inner product, so useful for *projections*.

$$\begin{aligned}
\mathbf{a}\wedge\mathbf{b} &= \tfrac{1}{2}\,a\,b\,((\mathbf{e}_1\,\mathbf{e}_1\cos\phi + \mathbf{e}_1\mathbf{e}_2\sin\phi) - (\mathbf{e}_1\,\mathbf{e}_1\cos\phi + \mathbf{e}_2\mathbf{e}_1\sin\phi))\\
&= a\,b\,\sin\phi\,\tfrac{1}{2}(\mathbf{e}_1\,\mathbf{e}_2 - \mathbf{e}_2\,\mathbf{e}_1) = a\,b\,\sin\phi\,(\mathbf{e}_1\wedge\mathbf{e}_2)
\end{aligned}$$

This is not a scalar or vector, but a *simple bivector*. It is a 2-dimensional area element. *The outer product spans subspaces.*

4 Complex numbers are real

Bivectors/area elements have negative squares. For example:

$$\begin{aligned}(\mathbf{e}_1 \wedge \mathbf{e}_2)\,(\mathbf{e}_1 \wedge \mathbf{e}_2) &= (\mathbf{e}_1\,\mathbf{e}_2)\,(\mathbf{e}_1\,\mathbf{e}_2) = \mathbf{e}_1\,(\mathbf{e}_2\,\mathbf{e}_1)\,\mathbf{e}_2 \\ &= \mathbf{e}_1\,(-\mathbf{e}_1\,\mathbf{e}_2)\,\mathbf{e}_2 = -(\mathbf{e}_1\,\mathbf{e}_1)\,(\mathbf{e}_2\,\mathbf{e}_2) = -1\end{aligned}$$

Let us call $\mathbf{e}_1 \wedge \mathbf{e}_2 \equiv \mathbf{i}$, with $\mathbf{i}\,\mathbf{i} = -1$, then:

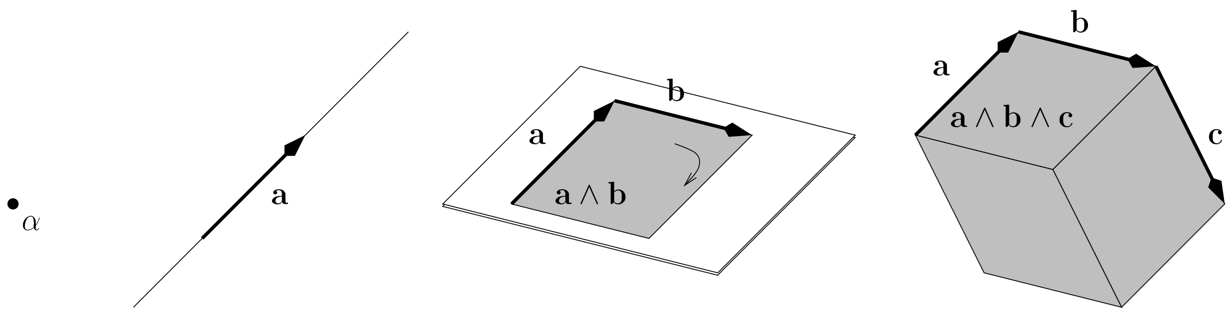
$$\mathbf{a}\,\mathbf{b} = a\,b\,(\cos\phi + \mathbf{i}\sin\phi) \overset{?!}{=} a\,b\,e^{\mathbf{i}\phi}$$

No wonder then that complex number systems are useful to geometry. But *bivectors are real*! Each real plane has its own 'complex number system' – so use it!

The argument $\mathbf{i}\phi$ is the *bivector angle* between $\mathbf{a}$ and $\mathbf{b}$. This is unique; no more annoying sign convention trouble in angle manipulations.

5 Computing directly with subspaces

We can extend the outer product associatively to span higher order subspaces.



Altogether, we get a basis of 2^m elements for all subspaces of any dimension in n-space. For 3-space:

$$\left\{ \underbrace{1}_{\textit{scalars}}, \quad \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}_{\textit{vector space}}, \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_3 \wedge \mathbf{e}_1}_{\textit{bivector space}}, \ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\textit{trivector space}} \right\}$$

Geometric algebra seems expensive... We'll get back to that.

Linear transformations f on the vector space 'lift' simply as *outermorphisms*:

$$f(\mathbf{a} \wedge \mathbf{b}) = f(\mathbf{a}) \wedge f(\mathbf{b}), \qquad \text{etc.}$$

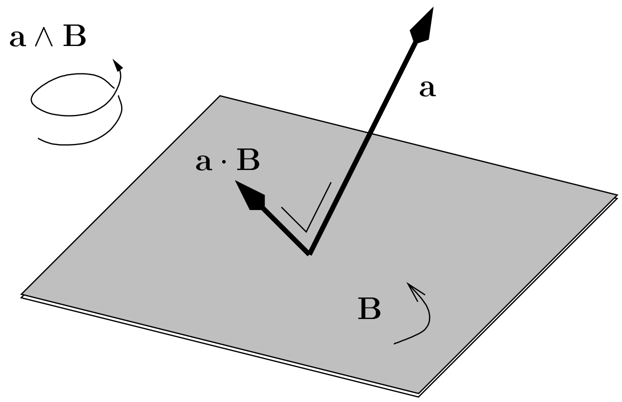
So: *'span of transformed is transform of spanned'*. Compare:

$$f(\mathbf{a} \times \mathbf{b}) = \overline{f}^{-1}(\mathbf{a}) \times \overline{f}^{-1}(\mathbf{b}) \ \det(f)$$

where $\overline{f}$ is the adjoint (or transpose). Ugly! More expensive! Avoid the cross product!

6 Orthogonality of subspaces

The inner product can similarly be extended to subspaces. $\mathbf{A} \cdot \mathbf{B}$ now means: the subspace of $\mathbf{B}$ perpendicular to $\mathbf{A}$ (More precisely: right-orthogonal in $\mathbf{B}$ to the projection of $\mathbf{A}$ onto $\mathbf{B}$).



This gives an algebra with equations like:

$$(\mathbf{A} \wedge \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$$

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \wedge (\mathbf{B} \cdot \mathbf{C}) \quad \text{if} \quad \mathbf{A} \subseteq \mathbf{C}$$

all geometrically meaningful in the interpretation of '$\wedge$' as span and '$\cdot$' as perpendicular part. Better than Boolean!

Especially frequent is taking the *dual* of an object, inner product with volume-element $\boldsymbol{\mathcal{I}}$. In 3D, this converts plane bivector into normal vector

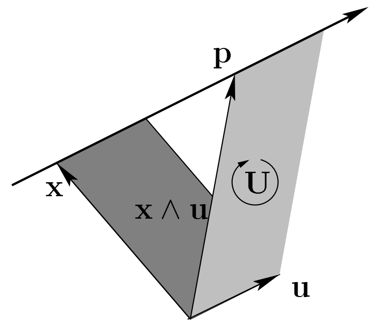
$$(\mathbf{p} \times \mathbf{q})\,\boldsymbol{\mathcal{I}}_3 = \mathbf{p} \wedge \mathbf{q}.$$

But bivectors are more general and much better behaved!

7 Familiar constructions remodeled

A line with direction $\mathbf{u}$ through a point $\mathbf{p}$ consists of all points $\mathbf{x}$ that span the same area with $\mathbf{u}$ as $\mathbf{p}$ does:

$$\mathbf{x} \wedge \mathbf{u} = \mathbf{p} \wedge \mathbf{u}.$$

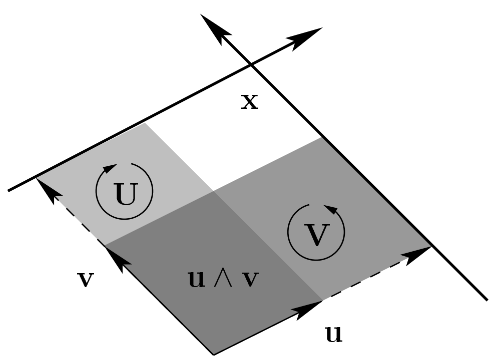


We call the bivector $\mathbf{U} \equiv \mathbf{p} \wedge \mathbf{u}$ the *moment* of the line; together with $\mathbf{u}$ it determines the line fully, in n-space! Note that $\mathbf{u} \wedge \mathbf{U} = 0$.

The solution to the line equation is $\mathbf{x} = \mathbf{p} + \lambda\mathbf{u}$, since:

$$(\mathbf{p} + \lambda\mathbf{u}) \wedge \mathbf{u} = \mathbf{p} \wedge \mathbf{u} + \lambda\, \mathbf{u} \wedge \mathbf{u} = \mathbf{p} \wedge \mathbf{u}.$$

Bivectors are reshapable!



Intersection of lines in the plane is easy once you 'see' bivectors:

$$\mathbf{x} = \frac{\mathbf{V}}{\mathbf{u} \wedge \mathbf{v}}\, \mathbf{u} + \frac{\mathbf{U}}{\mathbf{v} \wedge \mathbf{u}}\, \mathbf{v}$$

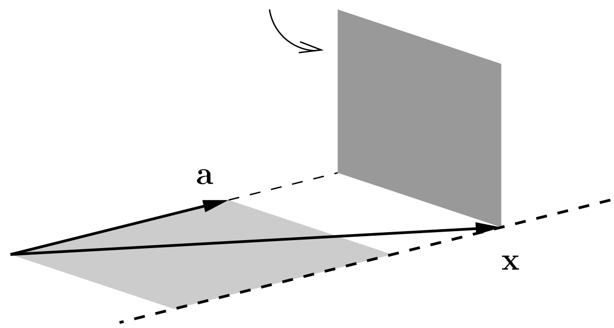
The geometrical contents of Kramer's rule.

8 Dividing by spaces: the invertibility of the GP

The inner product and outer product are not invertible separately.

> *Q: Given* $\mathbf{a}$ *and value of* $\mathbf{x} \cdot \mathbf{a}$*, what is* $\mathbf{x}$*?*
> *A: Somewhere on plane* $\mathbf{x} \cdot \mathbf{a} = \alpha$ *perpendicular to* $\mathbf{a}$*.*

> *Q: Given* $\mathbf{a}$ *and value of* $\mathbf{x} \wedge \mathbf{a} = \mathbf{A}$*, what is* $\mathbf{x}$*?*
> *A: Somewhere on the line parallel to* $\mathbf{a}$ *with moment* $\mathbf{A}$*.*



Combination of the two uniquely obviously defines $\mathbf{x}$; since the geometric product $\mathbf{x}\,\mathbf{a}$ equals $\mathbf{x} \cdot \mathbf{a} + \mathbf{x} \wedge \mathbf{a}$ contains both in a separable manner, it is invertible. This gives *division by vectors*!

$$\mathbf{x} = (\mathbf{x}\,\mathbf{a})/\mathbf{a} = \underbrace{(\mathbf{x} \cdot \mathbf{a})/\mathbf{a}}_{\textit{projection}} + \underbrace{(\mathbf{x} \wedge \mathbf{a})/\mathbf{a}}_{\textit{rejection}}$$

Division by $\mathbf{a}$ is multiplication by $\mathbf{a}^{-1}$, which is

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}$$

You can divide by subspaces of any dimension in geometric algebra.

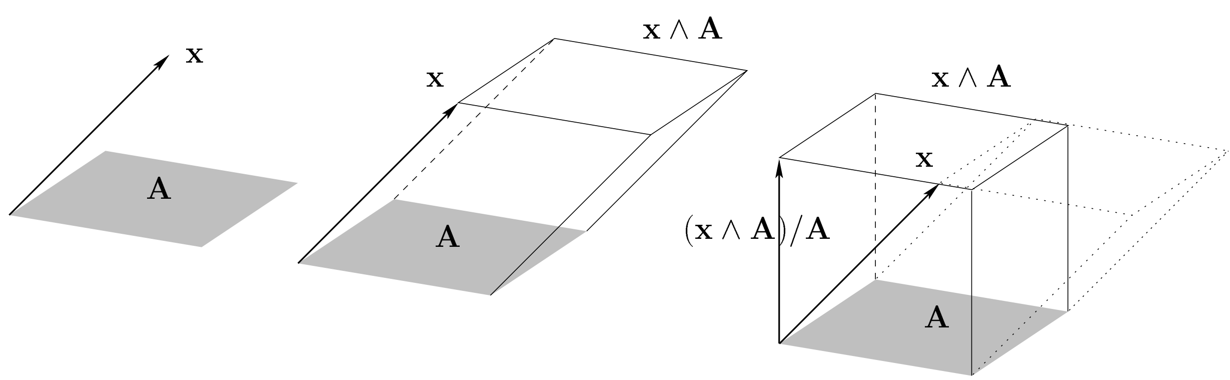
9 Projection and rejection

Let us try to find the perpendicular component $\mathbf{x}_\perp$ of a vector $\mathbf{x}$ to a plane with bivector $\mathbf{A}$ (both through the origin).

1. $\mathbf{x}_\perp$ is perpendicular to $\mathbf{A}$, so $\mathbf{x}_\perp \cdot \mathbf{A} = 0$.
2. $\mathbf{x}_\perp$ has the length to span the same volume as $\mathbf{x}$ with $\mathbf{A}$, so: $\mathbf{x}_\perp \wedge \mathbf{A} = \mathbf{x} \wedge \mathbf{A}$.
3. Combine these by adding them: $\mathbf{x}_\perp \mathbf{A} = \mathbf{x} \wedge \mathbf{A}$.
4. Divide by $\mathbf{A}$, this gives the answer

$$\boxed{\mathbf{x}_\perp = (\mathbf{x} \wedge \mathbf{A})/\mathbf{A}}$$

We call this the *rejection* of $\mathbf{x}$ by $\mathbf{A}$.



Similarly, the *projection* of $\mathbf{x}$ onto $\mathbf{A}$ is

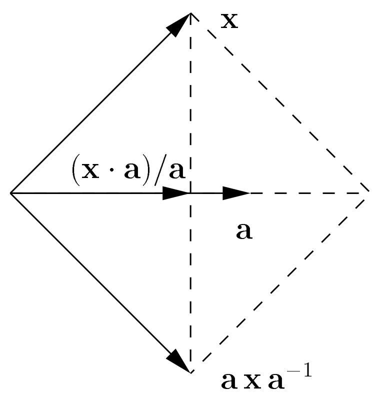
$$\boxed{\mathbf{x}_\parallel = (\mathbf{x} \cdot \mathbf{A})/\mathbf{A}}$$

This formula extends directly to the projection of a k-space onto an m-space within an n-space.

10 Reflection: objects as operators

The *reflection* of a vector $\mathbf{x}$ into a line characterized by the vector $\mathbf{a}$ is twice the projection minus $\mathbf{x}$:

$$2(\mathbf{a}\cdot\mathbf{x})\,\mathbf{a}^{-1}-\mathbf{x}=(\mathbf{a}\,\mathbf{x}+\mathbf{x}\,\mathbf{a})\,\mathbf{a}^{-1}-\mathbf{x}=\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}.$$



The line $\mathbf{a}$ acts as an operator on the line $\mathbf{x}$ by sandwiching:

$$\boxed{\mathbf{x}\;\;\mapsto\;\;\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}}$$

This is called a *versor product*. Useful for ray-tracing!

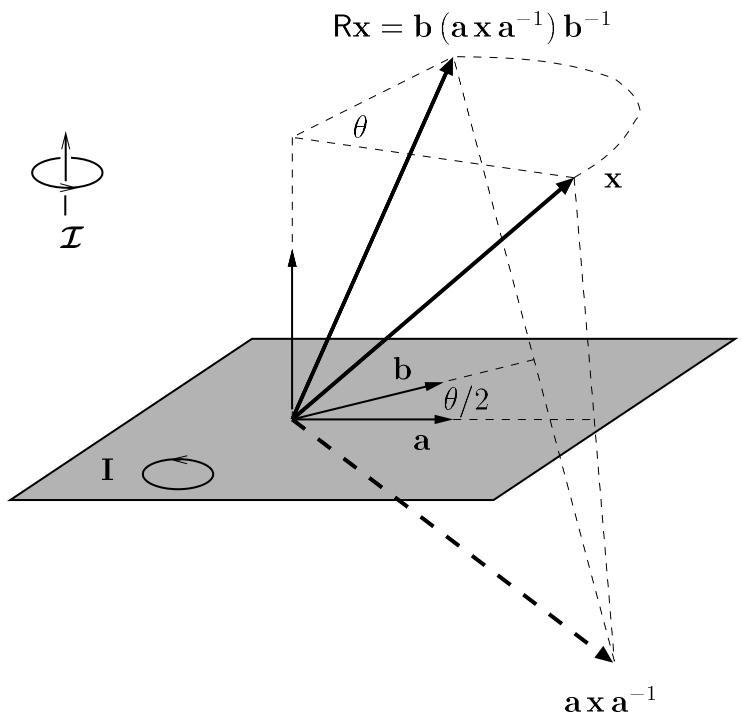
Reflection of a subspace $\mathbf{X}$ in a k-space $\mathbf{A}$ (both through the origin):

$$\boxed{\mathbf{X}\;\;\mapsto\;\;-(-1)^k\,\mathbf{A}\,\mathbf{X}\,\mathbf{A}^{-1}}$$

This sandwiching principle makes all geometrical *objects* in geometric algebra into geometrical *operators*.

11 Rotations as rotors (quaternions)

Two line reflections make a rotation over double the angle:



So with two vectors $\mathbf{a}$ and $\mathbf{b}$ the rotation in the $\mathbf{a} \wedge \mathbf{b}$ plane is:

$$\mathbf{x} \;\mapsto\; \mathbf{b}(\mathbf{a}\mathbf{x}\mathbf{a}^{-1})\mathbf{b}^{-1} = (\mathbf{b}\mathbf{a})\mathbf{x}\,(\mathbf{b}\mathbf{a})^{-1} = R\,\mathbf{x}\,R^{-1}$$

The *rotor* $R = \mathbf{b}\mathbf{a}$ characterizes the rotation fully.

We use these rotors as the representation of rotations. Take unit vectors for convenience, hence unit rotors.

$$R = \mathbf{b}\,\mathbf{a} = \cos(\theta/2) - \mathbf{I}\sin(\theta/2) = e^{-\mathbf{I}\theta/2},$$

where $\theta/2$ is the angle from $\mathbf{a}$ to $\mathbf{b}$, (and *half* the actual rotation angle) and $\mathbf{I}$ is the unit 2-blade for the $(\mathbf{a} \wedge \mathbf{b})$-plane.

12 Quaternions subsumed

Rotors are like quaternions, but embedded in algebra of real vectors.

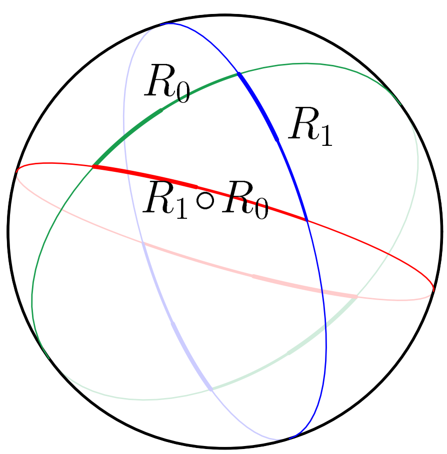
$$\text{quaternion } q_0 + \mathbf{q} \quad \leftrightarrow \quad \text{rotor } q_0 - \boldsymbol{\mathcal{I}}\mathbf{q}.$$

Here $\boldsymbol{\mathcal{I}}$ is the volume element of 3-dimensional space, and $\boldsymbol{\mathcal{I}}^2 = -1$. It makes duals: a rotation axis $\mathbf{q}$ into a rotation plane $\boldsymbol{\mathcal{I}}\mathbf{q}$.

The 'complex vector part' of a quaternion is actually a 'real bivector part'. All quaternion math then follows from geometric algebra.

$$\begin{aligned} qp &= (q_0 - \boldsymbol{\mathcal{I}}\mathbf{q})\,(p_0 - \boldsymbol{\mathcal{I}}\mathbf{p}) \\ &= q_0 p_0 + \langle \boldsymbol{\mathcal{I}}\mathbf{q}\boldsymbol{\mathcal{I}}\mathbf{p}\rangle_0 - \boldsymbol{\mathcal{I}}(\mathbf{q}p_0 + \mathbf{p}q_0 - \boldsymbol{\mathcal{I}}^{-1}\langle \boldsymbol{\mathcal{I}}\mathbf{q}\boldsymbol{\mathcal{I}}\mathbf{p}\rangle_2) \\ &= q_0 p_0 - \langle \mathbf{q}\mathbf{p}\rangle_0 - \boldsymbol{\mathcal{I}}(\mathbf{q}p_0 + \mathbf{p}q_0 + \boldsymbol{\mathcal{I}}^{-1}\langle \mathbf{q}\mathbf{p}\rangle_2) \\ &= p_0 q_0 - \mathbf{p}\cdot\mathbf{q} + \boldsymbol{\mathcal{I}}(p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{q}\times\mathbf{p}), \end{aligned}$$

well-known in quaternion literature, but somewhat *ad hoc* there.



Linear interpolation of rotations:

$$R(\lambda) = \frac{\sin((1-\lambda)\frac{\theta}{2})}{\sin(\frac{\theta}{2})}\, R_0 + \frac{\sin(\lambda\frac{\theta}{2})}{\sin(\frac{\theta}{2})}\, R_1$$

Quaternions are very real – not involving 'complex vectors' but 'real bivectors'.

13 `meet` and `join`: the incidence products

The incidence between two subspaces $\mathbf{A}$ and $\mathbf{B}$ is called their `meet`.

$$\mathbf{A} \cap \mathbf{B} = (\mathbf{B}/\mathbf{J}) \cdot \mathbf{A}$$

where $\mathbf{J}$ is the `join`, the smallest common superspace of $\mathbf{A}$ and $\mathbf{B}$. This gives a quantitative incidence measure – common subspace of two planes is 'strongest' when they are orthogonal.

Dually in $\mathbf{J}$ (denoting $\mathbf{X}^* \equiv \mathbf{X}/\mathbf{J}$):

$$(\mathbf{A} \cap \mathbf{B})^* = \mathbf{B}^* \wedge \mathbf{A}^*.$$

Much generalizes the useful and compact 'cross product of normal vectors' construction for plane intersection, to arbitrary subspaces of arbitrary spaces.

14 Models of geometry: homogeneous model

Geometric algebra gives the basic operations of geometry. By casting spaces into the algebra, this can give great power.

Homogeneous model: use 1 extra dimension e for the origin.
A point at $\mathbf{p}$ becomes a vector p;

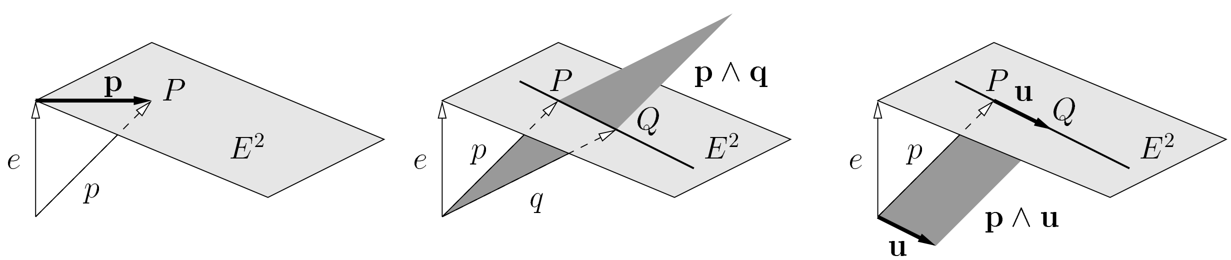
$$\textit{point } P \textit{ at } \mathbf{p} \iff p = e + \mathbf{p}$$

Then it is simple to go to higher order subspaces using the span:

$$\textit{line element through } P \textit{ and } Q \iff p \wedge q$$

Explicitly:

$$\begin{aligned} p \wedge q &= (e + \mathbf{p}) \wedge (e + \mathbf{q}) = e \wedge (\mathbf{q} - \mathbf{p}) + \mathbf{p} \wedge \mathbf{q} \\ &= e \wedge (\mathbf{q} - \mathbf{p}) + \mathbf{p} \wedge (\mathbf{q} - \mathbf{p}) = e \wedge \underbrace{\mathbf{u}}_{\textit{direction}} + \underbrace{\mathbf{p} \wedge \mathbf{u}}_{\textit{moment}} \end{aligned}$$



A general *plane* is a trivector $p \wedge q \wedge r$, or dually a vector $\mathbf{n} - \delta e$.

15 Plücker incorporated

The 6-parameter line representation could be familiar:

$$p \wedge q = e \wedge (\mathbf{q} - \mathbf{p}) + \mathbf{p} \wedge \mathbf{q} = e\,(\mathbf{q} - \mathbf{p}) + (\mathbf{p} \times \mathbf{q})\,\boldsymbol{\mathcal{I}}$$

This is really just the 6 Plücker coordinates $[\mathbf{q} - \mathbf{p}, \mathbf{p} \times \mathbf{q}]$.

All Plücker incidence formulas follow automatically using the `meet`. For instance, a line ℓ and a plane π intersecting in a point:

$$\pi^* \cdot \ell = (\mathbf{n} - \delta e) \cdot (e\mathbf{u} + \mathbf{m}\boldsymbol{\mathcal{I}}_3) = -(\mathbf{n} \cdot \mathbf{u})e + (\mathbf{m} \times \mathbf{u} - \delta \mathbf{u})$$

Two lines in 3D having their orthogonal distance as the incidence:

$$\ell_2{}^* \cdot \ell_1 = (-\mathbf{u}_2\boldsymbol{\mathcal{I}}_3 + \mathbf{m}_2 e) \cdot (e\mathbf{u}_1 + \mathbf{m}_1\boldsymbol{\mathcal{I}}_3) = \mathbf{m}_2 \cdot \mathbf{u}_1 + \mathbf{m}_1 \cdot \mathbf{u}_2$$

Plücker coordinates are simply part of doing homogeneous coordinates consistently – in GA. Not as arcane as some think!

Quaternions, Plücker, hmm. GA appears capable of naturally giving most efficient computational representations found for operations – even though it is hi-D.... This could go somewhere.

16 Models of geometry: double homogeneous model

By going two dimensions up, so embedding 3-space in 5-space and using its 2^5-dimensional algebra, things become very nice.

The extra dimensions are the origin e and the point at infinity e_∞. A vector now represents a *dual sphere*. The inner product immediately gives the Euclidean distance

$$p \cdot q = -\tfrac{1}{2}\, d_E^2(p, q)$$

So a *point* is represented by a *null vector* p with $p\,p = 0$:

$$p = \mathbf{p} + e - \tfrac{1}{2}\mathbf{p}^2 e_\infty$$

Now the outer product spans *spheres*:

$$\textit{1-sphere (ordered point pair)}\ (p, q) \iff p \wedge q$$

$$\textit{2-sphere (circle) through}\ p, q, r \iff p \wedge q \wedge r$$

et cetera. Flat subspaces are spheres containing e_∞:

$$\textit{line element through}\ p, q \iff e_\infty \wedge p \wedge q$$

$$\textit{plane element through}\ p, q, r \iff e_\infty \wedge p \wedge q \wedge r$$

et cetera.

17 Double homogeneous (conformal) model - 2

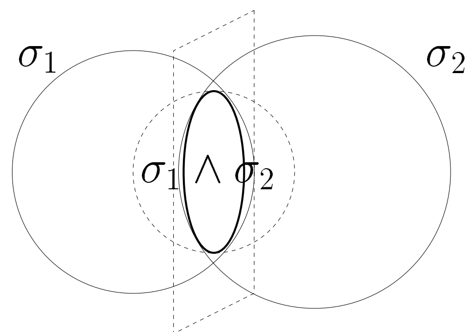
We can now use all GA operations directly on spheres and planes.

- Midplane between two points p and q is dually $q - p$.
- Radius and center of sphere through orthogonal complement:

$$(p \wedge q \wedge r \wedge s)^* = c + \tfrac{1}{2}\rho^2 e_\infty$$

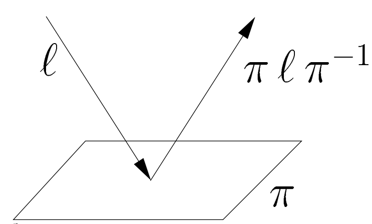
- Intersect two spheres through the `meet`:

$$\underbrace{\sigma_1 \wedge \sigma_2}_{circle} = \underbrace{\frac{\sigma_1 \wedge \sigma_2}{\sigma_2 - \sigma_1}}_{perp.\ sphere} \wedge \underbrace{(\sigma_2 - \sigma_1)}_{int.\ plane}$$



- Reflect line ℓ in plane π:

$$\ell \;\mapsto\; \pi\,\ell\,\pi^{-1}.$$



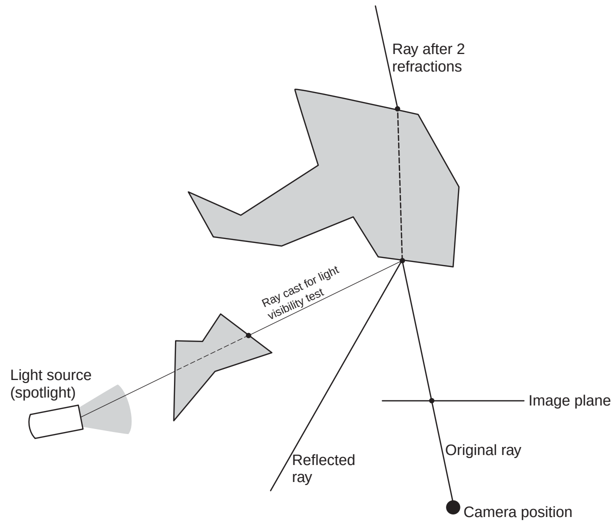
This is a *full specification* of the line.

- Conformal mappings on object A (incl. rigid body motions):

$$A \;\mapsto\; R\,A\,R^{-1}, \quad \text{with } R = e^{\mathbf{B}} \text{ a 10-component versor}$$

Planes and spheres are objects or operators, all is unified. All operations are defined in a coordinate-free manner.

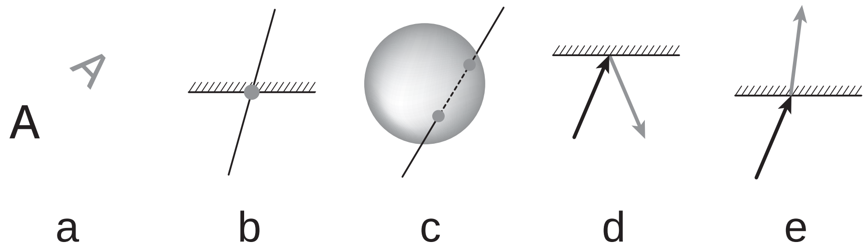
18 A ray tracer



(picture generated using the 5D GA of GAIGEN)

19 Relative performance of various algebras

A ray tracer requires some basic 3D geometry capabilities:



(a) rotation/translation

(b) intersect point and line

(c) intersect line and sphere

(d) reflect line in plane

(e) refract line through plane (Snell's law)

We can do this using many different models of geometry – same algorithms, different algebra for basic operations.

model	impl.	rendering
3D LA	ad hoc	$1.0\times$
3D GA	GAIGEN	$3.0\times$
4D LA	ad hoc	$0.98\times$
4D GA	GAIGEN	$3.0\times$
5D GA	GAIGEN	$5.8\times$

Presumably 5D GA can be within $1.5\times$ to $3\times$ 3D LA. This may be an acceptable price for a much simpler programming structure.

20 Example: Snell's law using 5D GA

Remember? Requires intersection of line (ray) and plane, expensive sines of the correct angles in this plane:

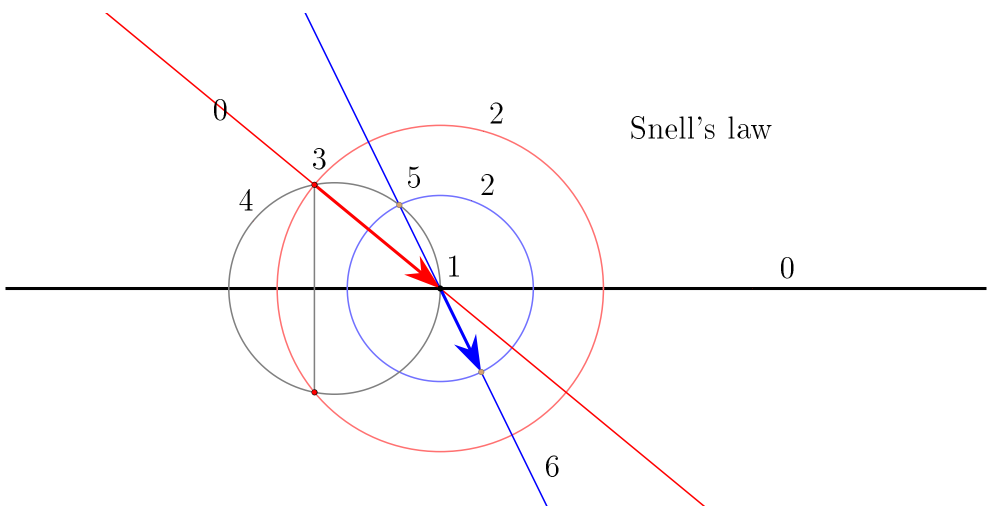
$$\frac{\sin\alpha_i}{\sin\alpha_o} = \frac{v_i}{v_o} \equiv \nu$$

then rotation of ray over the new angle, in plane of ray and normal of boundary plane, around intersection point. Or write it out:

$$\mathbf{u} \quad\mapsto\quad \left(\mathrm{sign}(\mathbf{n}\cdot\mathbf{u})\sqrt{(1-\nu^2)\,\mathbf{u}^2\mathbf{n}^2 + \nu^2(\mathbf{n}\cdot\mathbf{u})^2} - \nu\,(\mathbf{n}\cdot\mathbf{u})\right)\mathbf{n}^{-1} + \nu\,\mathbf{u}$$

$$\mathbf{p} = \frac{\delta\mathbf{u} - \mathbf{n}\cdot\mathbf{U}}{\mathbf{n}\cdot\mathbf{u}}$$

Construction with 3D spheres, a few basic products on 5D vectors:



1. line intersect plane ($1\wedge$)
2. span spheres $\mathbf{c}, \rho$: ($2+$)
3. line intersect sphere ($1\wedge$)
4. span perpendicular circle ($1\wedge$)
5. intersect circle and sphere ($1\wedge$)
6. connect points ($1\wedge$)

21 Conclusions

- Geometric Algebra works!
- It has much more structure than the classical hotchpotch.
- It contains known efficient representations of operations and objects, (rotations, `meet`, Plücker), much extends them, and begins to add new ones.
- It will eliminate dimension-dependent cases in geometric software, thus reducing bugs.
- Efficient implementations are being made.
- Big 'But…' – we have to learn how to wield it.

Want to know more? Visit

`www.science.uva.nl/~leo/ga/`

This has software, tutorial introductions, links to the community.